Charlie Nusbaum
Econ 204B
February 9, 2018

Section 4 Notes: Pop-Quiz Review and Recursive Competitive Equilibrium

Pop Quiz: Consider a consumer who derives utility from both nondurable consumption, c_t, and a flow of serviecs from durables. The stock of durables evolves according to

$$d_{t+1} = (1-\delta)d_t + e_t$$

where δ is depreciation and e_t is the purchase of durables. Consumers may buy or sell durables so e_t is not bounded below by 0. Purchased durables don't yield any services until tomorrow. Preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, d_t) \qquad 0 < \beta < 1$$

Each period, consumers receive exogenous income y_t. Income is i.i.d. and takes I values, each with probability π_i. Consumers can also save assets. The budget constraint is thus given by

$$c_t + e_t + a_{t+1} = y_t + (1+r)a_t$$

1. Formulate the Bellman equation for this problem:

$$V(a,d,y) = \max_{a',d'} \{U(c,d) + \beta \sum_i V(a',d',y_i)\pi_i\}$$

$$s.t. \qquad c + e + a' = y + (1+r)a$$

$$d' = (1-\delta)d + e$$

$$c, d' \geq 0 \;\; a' \geq \underline{a}$$

2. Modify the problem to incorporate the assumption that durables purchased today yield services immediately. That is, they immediately enter your stock of durables.

$$V(a,d,y) = \max_{a',d'} \{U(c,d+e) + \beta \sum_i V(a',d',y_i)\pi_i\}$$

$$s.t. \qquad c + e + a' = y + (1+r)a$$

$$d' = (1-\delta)(d+e)$$

$$c, d' \geq 0 \;\; a' \geq \underline{a}$$

3. Now suppose that the investment in consumer durables is partially irreversible in the sense that durables sell at a discount. In particular, the price of buying durables is equal to 1 as before, but durables can be sold at price $p < 1$. Formulate the Bellman equation.

$$V(a,d,y) = \max\{V^b(a,d,y), V^s(a,d,y)\}$$

where,

$$V^b(a,d,y) = \max_{a',e\geq 0,d'} \{U(c,d+e) + \beta \sum_i V(a',d',y_i)\pi_i\}$$

$$s.t. \qquad c + e + a' = y + (1+r)a$$

$$d' = (1-\delta)d + e$$

and,

$$V^s(a,d,y) = \max_{a',e\geq 0,d'} \{U(c,d+e) + \beta \sum_i V(a',d',y_i)\pi_i\}$$

$$s.t. \qquad c - pe + a' = y + (1+r)a$$

$$d' = (1-\delta)d - e$$

Stochastic Dynamic Programming

Suppose that we have a standard optimal growth problem that is augmented with TFP shocks:

$$U = \max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad\qquad c_t + k_{t+1} = z_t f(k_t)$$

$$\Leftrightarrow U = \max \sum_{t=0}^{\infty} \sum_{z \in Z} \beta^t u(z_t f(k_t) - k_{t+1}(z_{t+1}))$$

Notice what has changed. z_t changes the choice set for individuals and is thus relevant for the individual's decision problem. As a result, we have added a second state variable! Now, let's formulate the Bellman equation:

$$V(k, z) = \max_{0 \geq k' \geq z f(k)} \{u(z f(k) - k') + \beta \mathbb{E}(V(k', z')|z)\}$$

Notice that z affectss two things:

1. The choice set (correspondence), $\Gamma(\cdot)$
2. The objective function

The combination of these two things force us to make additional assumptions to obtain the value function properties we discussed last week.

Existence and Uniqueness
To maintain the existence and uniqueness properties, we must add an assumption regarding the correspondence and the expectations operator. In particular:
(A1): Γ must be continuous in z
(A2): $\mathbb{E}(\cdot)$ maps continuous functions onto continuous functions

Monotonicity
To maintain monotonicity of the value function, we need to modify one assumption on the correspondence and the expectations operator. In particular:
(A3): Γ is monotone $\forall\ x, z$
(A4): Π is monotone: If $f'(z) > 0$, then $\hat{f}(z) = \sum_{z \in Z} f(z)\Pi(z'|z)$ is also increasing in z

Concavity/Differentiability To maintain concavity and differentiability, we need to modify several assumptions as before. In particular:
(A5): U is jointly concave in (x, z)
(A6): Γ is convex in x $\forall\, z$
(A7): U is differntiable in x $\forall\, z$
(A8): $g(x, z)$ is on the interior of $\Gamma(x, z)$

Recursive Competitive Equilibrium

So far, we have not endogenized prices in our recursive framework: we've only been solving partial equilibrium models.

Household's Problem:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad s.t. \qquad c_t + k_{t+1} \leq w_t + (r_t + 1 - \delta)\ \forall t \geq 0$$

$$k_{t+1}, c_t \geq 0$$

Firm's Problem:

$$\Pi = \max_{K_t, H_t} \{F(K_t, H_t) - w_t H_t - r_t K_t\}$$

Note that under the assumption of constant returns to scale production, firms earn 0 profits in equilibrium.

Market Clearing: Need aggregate supply and demand for capital and labor to be equal and the aggregate resource constraint to hold.

Sequential Markets Equilibrium
A sequential markets equilibrium is an allocation $(c_t, k_{t+1}^s, Y_t, K_t^d, H_t^d)_{t=0}^{\infty}$ and prices $(w_t, r_t)_{t=0}^{\infty}$ such that

i. $(c_t, k_{t+1}^s)_{t=0}^{\infty}$ solves the household's problem taking prices as given

ii. $(K_t^d, H_t^d)_{t=0}^{\infty}$ solves the firm's problem taking prices as given

iii. $(w_t, r_t)_{t=0}^{\infty}$ are such that markets clear:

$$K_t^d = K_t^s$$
$$H_t^d = 1$$
$$Y_t = C_t + K_{t+1}^s - (1-\delta) K_t^s$$

Notice that the sequential markets equilibrium maintains our need to solve for sequences. Turning to the recursive competitive equilibrium, we can make our job easier by simply solving for functions.

Recursive Household Problem

$$V(k,K|G)=\max_{c,k'}\{u(c)+\beta V(k',K'|G)\} \qquad s.t. \qquad c+k'\leq w(K)+(1+r(K)-\delta)k$$

$$k',c\geq 0$$

$$K'=G(K)$$

$$k'=g(k,K|G)$$

Here, G are the household's *arbitrary* expectations over the law of motion for aggregate capital. Moreover, notice that households must track aggregate capital to keep track of prices.

Recursive Competitve Equilibrium

A recursive competitive equilibrium given by arbitrary expectations, G, consists of a value function $V(k,K|G)$, a policy function, $g(k,K|G)$, and pricing functions, $r(K)$ and $w(K)$, such that

i. $V(k,K|G)$ and $g(k,K|G)$ solves the household problem taking G, r, and w as given.

ii. $r(K)=F_K(K,1)$ and $w(K)=F_H(K,1)$

iii. $K'=g(K,K|G)=\tilde{H}(K|G)$

Additionally, the aggreagte resource constraint must be satisfied. Note that the second and third conditions ensure that individual behavior is consistent with aggregate outcomes. Moreover, condition 3 doesn't ensure that expectations are rational in the sense that these expectations are not necessarily consistent with aggregate behavior.

Rational Expectations Recursive Competitive Equilibrium

A RERCE consists of a value function $V(k,K|G^*)$, a policy function $g(k,K|G^*)$, expectations over the aggregate law of motion for capital, $G^*(K)$, and pricing functions, $r(K)$ and $w(K)$, such that

i. $V(k,K|G^*)$ and $g(k,K|G^*)$ solves the household's problem given expectations and prices

ii. $r(K)=F_K(K,1)$ and $w(K)=F_H(K,1)$ taking prices and expectations as given

iii. $K'=g(k,K|G^*)=\tilde{H}(K|G^*)$

iv. $G^*=\tilde{H}(K|G^*)$

Notice that condition 4 ensures that individual have rational expectations. That is that individual expectations are also consistent with aggregate behavior. Again, the resource constraint must also be satisfied.

Example

Suppose that we augment the standard Neo-Classical growth model with a public sector. A government levies taxes on capital gains to fund a fixed level of government spending, P.

<u>Household's Problem</u>

$$V(k,K|G,\tau) = \max_{c,k'}\{u(c) + \beta V(k',K'|G,\tau)\} \qquad s.t. \qquad c + k' \leq w(K) + \tau(K)r(K)K + (1-\delta)k$$

$$k', c \geq 0$$

$$K' = G(K)$$

$$k' = g(k,K|G,\tau)$$

Rational Expectations Recursive Competitive Equilibrium
Given, P, consists of a value function $V(k,K|G,\tau)$, a policy function $g(k,K|G,\tau)$, pricing functions $r(K)$ and $w(K)$, expectations over the aggregate law of motion $G^*(K)$, and a tax function $\tau(K)$, such that

i. $V(k,K|G^*,\tau)$ and $g(k,K|G^*,\tau)$ solve the household's problem taking prices and taxes as given

ii. $r(K) = F_K(K,1)$ and $w(K) = F_H(K,1)$

iii. $K' = g(K,K|G^*,\tau) = G^*(K)$

iv. $P = \tau(K)r(K)K$

v. Markets clear

We've imposed market clearing in condition 2, but it's good practice to write it out explicitly along with the aggregate resource constraint: $Y = C + G + K' - (1-\delta)K$ where $K' - (1-\delta)K$ is aggregate savings.